Error Minimization Using Redundant Gravity Measurements

Igor Kulikov and Michail Zak[1)]

Exploration Systems Autonomy Section 367
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109

Abstract

This paper addresses the problem of redundant gravity measurements for reduction of measurement errors. The approach exploits constraints imposed upon the components of the gravity gradient tensor by the conditions of integrability needed for reconstruction of the gravity potential. It has been demonstrated that the total error of noisy measurements can be reduced by 25% using the best fit into the integrability constraints.

1. INTRODUCTION

Recent success in the development of a unique and innovative technology of atom trapping and laser cooling in combination with atom intereferometry has opened up a fundamentally new approach to detection of the rotations and accelerations, and has led to the success in the development of matter wave gravimeters and gradiometers (Adams, 1994; Kasevich and Chu, 1992; Snadden et al., 1999)

Use of highly accurate and space born gravity measuring devices opens a new era in geophysics and space exploration (Chan and Paik, 1992). It can successfully facilitate space reconnaissance operations for the detection of suspicious objects on the ground and related intelligence activities needed for global information superiority (Bell, 1998).

However, despite the unprecedented accuracy of the new class of gradiometers, there is still a number of sources of noise such as atmospheric drag, inertial noise, attitudinal error, induced gravity noise, Eötvos-effect induced noise etc (Peters et al., 1999; Clauser, 2000; Young et al., 1997). Obviously noisy measurement decreases the effectiveness of the whole approach. Therefore, minimization of noisy errors is one of the main problems in this new technology.

In this paper, we propose a methodology (illustrated by the proof-of-concept example) for reduction of the total error of gravity gradient tensor using redundant measurements. For that purpose we will start with addressing the following problem: what is the degree of redundancy in

[1)] kulikov@jpl.nasa.gov
mzak@jpl.nasa.gov

values of the gravity gradient tensor? In other words, what is the minimum of necessary measurements of the components of this tensor which allows one to reconstruct the gravitational potential? The problem arises from the fact that gravity gradient tensor is derived from a scalar potential of gravitational field (Parker, 1994). Therefore, the components of gravity gradient tensor cannot change independently: they must satisfy the integrability conditions which impose geometrical constraints upon their changes. Thus the purpose of the second section is to formulate these constraints. Based upon them, one can develop an optimal strategy for the measurements of the gravity gradient tensor which includes the decisions about how many measurements of each component can be ignored without loss of information about the potential and how the redundant measurements can decrease an error in the reconstructed potential. These problems will be analyzed in the third section. In the fourth section, we will formulate the methodology for reduction of the total error by the best fit into the integrability constraints. The fifth section will be devoted to a proof-of-concept example and the results of computer simulations.

2. INTEGRABILITY CONSTRAINTS

General formalism

We will start with a brief review of the gravitational field formalism. A gravitational potential $\varphi(\vec{x})$ at the point $\vec{x}$ created by a mass distribution comprised of N point masses m_i respectively, and located at the positions $\vec{x}_i$ is

$$\varphi(\vec{x}) = -G\sum_{i=1}^{N}\frac{m_i}{|\vec{x}-\vec{x}_i|} \tag{1}$$

where G is constant of gravitational interaction. The first gravity gradient associated with $\varphi(\vec{x})$ forms a potential vector field

$$\vec{g}(\vec{x}) = -\nabla\varphi(\vec{x}) \tag{2}$$

The second gravity gradient (gravity gradient tensor) induced by the vector field (2) in invariant form is

$$\Gamma(\vec{x}) = -\nabla g(\vec{x}) = \nabla\nabla\varphi(\vec{x}) \tag{3}$$

Coordinate representations for the first $g_\alpha(\vec{x}) = -\partial_\alpha \varphi(\vec{x})$ and the second $\Gamma_{\alpha\beta}(\vec{x}) = -\partial_\alpha g_\beta(\vec{x})$ gravity gradients are

$$
\begin{aligned}
g_\alpha(\vec{x}) &= -G\sum_{i=1}^{N} m_i \frac{(\vec{x}-\vec{x}^i)_\alpha}{|\vec{x}-\vec{x}^i|^3}, \\
\Gamma_{\alpha\beta}(\vec{x}) &= -G\sum_{i=1}^{N} m_i \frac{3(\vec{x}-\vec{x}^i)_\alpha(\vec{x}-\vec{x}^i)_\beta - \delta_{\alpha\beta}|\vec{x}-\vec{x}^i|^2}{|\vec{x}-\vec{x}^i|^5}
\end{aligned}
\tag{4}
$$

Zero trace of gravity gradient tensor follows from the identity

$$\nabla(\nabla\varphi(\vec{x})) = \Delta\varphi(\vec{x}) \equiv 0 \tag{5}$$

representing the fact that the gravity field does not have sources or sinks, and therefore its Laplacian is zero.

First gradient constraints

Let us first analyze the integrability constraints imposed upon the components of the first gradient (2). The origin of these constraints comes from the fact that three components of the vector of gravitational acceleration are derived from one scalar $\varphi(\vec{x})$. Therefore, they cannot be independent. Two additional constraints must be imposed upon them. The integrability constraints follow from the identity

$$\nabla \times \vec{g}(\vec{x}) = \nabla \times \nabla\varphi(\vec{x}) \equiv 0 \tag{6}$$

and can be rewritten in Cartesian coordinates as

$$\partial_2 g_1 = \partial_1 g_2 = \partial_1\partial_2\varphi,\ \partial_3 g = \partial_1 g_3 = \partial_1\partial_3\varphi,\ \partial_3 g_2 = \partial_2 g_3 = \partial_2\partial_3\varphi \tag{7}$$

These equations indicates the symmetry of gravity gradient tensor which can be written as

$$\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha} \tag{8}$$

Now the physical meaning of the integrability constraints (7) becomes clear: they enforce equalities of the mixed derivatives of the gravity potential $\varphi(\vec{x})$ with respect to the coordinates $\{x_i\}$, i.e., they provide the conditions for integrability which allow one to reconstruct $\varphi(\vec{x})$ from the components of the first gradient (2). However, the constraints (7) are not independent since they are bounded by the following scalar identity

$$\nabla \cdot (\nabla \times \vec{g}(\vec{x})) \equiv 0 \tag{9}$$

Thus, only two constraints out of three in equations (7) are truely independent, and therefore, the gravity gradient vector has only one degree of freedom, and that is consistant with the fact that the gradient is derived from a scalar $\varphi(\vec{x})$. The identity (9) can be interpreted as a vector version of the Bianchi identities, which bound the integrability constraints. As in tensor analysis, equation (9) involves higher order space derivatives; this means that in an initially chosen point of space, x_1^0, x_2^0, x_3^0, the constraints (7) are independent; however, as we move to the neighboring points, the changes of these constraints become correlated via equation (9).

Second gradient constraints

The integrability constraints for the second gradient follow from the identity (Zak, 1994):

$$\nabla \times \Gamma(\vec{x}) = \nabla \times (\nabla \vec{g}(\vec{x})) \equiv 0 \tag{10}$$

and can be written in the Cartesian coordinates as

$$\begin{aligned}
\partial_2 \Gamma_{31} = \partial_3 \Gamma_{21},\ \partial_2 \Gamma_{32} = \partial_3 \Gamma_{22},\ \partial_3 \Gamma_{33} = \partial_3 \Gamma_{23} \\
\partial_3 \Gamma_{11} = \partial_1 \Gamma_{31},\ \partial_3 \Gamma_{12} = \partial_1 \Gamma_{32},\ \partial_3 \Gamma_{13} = \partial_1 \Gamma_{33} \\
\partial_1 \Gamma_{21} = \partial_2 \Gamma_{11},\ \partial_1 \Gamma_{22} = \partial_2 \Gamma_{12},\ \partial_1 \Gamma_{23} = \partial_2 \Gamma_{13}
\end{aligned} \tag{11}$$

These nine integrability constraints are not independent because of the following vector identity

$$\nabla \cdot (\nabla \times \Gamma(\vec{x})) \equiv 0 \tag{12}$$

which bounds the integrability constraints in the same way as equation (9) bounds the constraints (8). Equations (11) and (8) form a complete system of twelve integrability constraints. However, because of the Bianchi identities (9) and (12), only eight of them are independent. Therefore, nine components of the gravity gradient tensor (3) are bound by eight constraints, and the gravity gradient tensor has only one degree of freedom which, again, is consistent with the fact that this tensor is derived from a scalar $\varphi(\vec{x})$. As in the case of the gravitational vector, the constraints (8) and (11) are independent in an initially chosen point of space, however, as we move to the neighboring points, the changes of these constraints become correlated via equations (9) and (12). The geometrical meaning of the integrability constraints (11) is the same as for the constraints (8): they result from the equality of the mixed derivatives $\partial_\alpha \partial_\beta \partial_\gamma \varphi = \partial_\beta \partial_\alpha \partial_\gamma \varphi$, etc., which allows one to reconstruct the gravity potential φ from the components of gravity gradient.

Finally, the twelve integrability constraints (8) and (11) must be supplemented by the constraint (5) which can be rewritten in the form

$$\delta^{\alpha\beta}\Gamma_{\alpha\beta} = 0 \tag{13}$$

One should recall that this is not an integrability constraint: it follows from the absence of sources and sinks in the gravity field.

3. MEASUREMENT STARATEGY

One of the advantages of state-of-the-art gradiometers is that they can measure the components of the gravity gradient tensor (3), which are the most sensitive to changes of mass densities. Now one can ask the following question: what is the minimum number of measurements which is sufficient for reconstruction of the gravity potential, and thereby the rest components of the gravity gradient tensor, and how the redundant measurements can help to reduce errors of noisy measurements? We will start with the first part of this question. Let us assume that one has to reconstruct gravity gradient tensor in n^3 points of a cube. If n^3 is sufficiently large, one can come up with a simple asymptotic estimate for the minimum number of measurements N

$$N \to n^3 \; if \; n^3 \to \infty \tag{14}$$

Indeed, as shown in the previous section, the gravity gradient tensor has only one degree of freedom, and therefore, only one component of the tensor at each point is sufficient while the rest $8n^3$ components are found from the constraints (8) and (11). However, since the constraints (11) are differential, some additional measurements (which can be associated with initial conditions) are required. For that purpose, suppose that x_1^0, x_2^0, x_3^0 is the initial point. Then one has to make five measurements to reconstruct all the nine components of the gravity gradient tensor, for instance

$$\begin{aligned}&\Gamma_{11}\left(x_1^0,x_2^0,x_3^0\right),\ \Gamma_{22}\left(x_1^0,x_2^0,x_3^0\right),\ \Gamma_{12}\left(x_1^0,x_2^0,x_3^0\right)\\ &\Gamma_{13}\left(x_1^0,x_2^0,x_3^0\right),\ \Gamma_{23}\left(x_1^0,x_2^0,x_3^0\right)\end{aligned} \tag{15}$$

The rest components are easily found from the algebraic constraints (8) and (13). If the same measurements (15) are performed in n points along the x_1, one finds the gravity gradient tensor $\Gamma_{ij}\left(x_1^0,x_2^0,x_3^0\right)$ and all its derivatives $\partial_1\Gamma_{\alpha\beta}$. Then, based upon the intergability constraints (11), one can reconstruct Γ_{11},Γ_{12} and Γ_{13} along the line $x_1^0, x_2^0+\Delta x_2^0, x_3^0$. After two additional measurements, for instance, Γ_{22} and Γ_{23}, at each point of the same line and with the help of the constraints (8) and (13), one can reconstruct the gravity gradient tensor along this line. Continuing this process by moving toward new points on the plane x_1^0, x_2^0, x_3^0, one reconstructs the tensor-gradient at each point of this plane. This process will require $N' = 5n + 2n(n-1)$ measurements. Finally, at the rest $n^2(n-1)$ points one has to measure only one component of the gravity gradient tensor, which is $\partial_3\Gamma_{33}$. Indeed, moving forward along the axis x_3 starting with x_3^0, and involving the constraints (8), (11) and (13), one finds the rest of the components in all the volume n^3. Thus, out of $9n^3$ components of the gravity gradient tensor, only

$$\begin{aligned}N &= 5n + 2n(n-1) + n^2(n-1)\\ &= n^3 + n^2 + 3n\end{aligned} \tag{16}$$

measurements are required. Obviously $n^3+n^2+3n \to n^3$ *if* $n \to \infty$ i.e., one returns to the equation (14).

Another advantage of exploiting the integrability constraints is in reducing the total error of noisy measurements by making redundant measurements. Indeed, as follows from equation (16), if the number of performed measurements $N_0 > n^3 + n^2 + 3n$, then there are $N_0 - n^3 - n^2 - 3n$ redundant measurements, and they can be used for reducing the noisy measurements in the following way. Let us introduce the following function to be minimized

$$M = \frac{1}{2}\sum_{i=1}^{n^3}\sum_{\alpha\beta}\left(\Gamma_{\alpha\beta}^{*i} - \Gamma_{\alpha\beta}^{i}\right)^2 \tag{17}$$

subject to the constraints:

$$f_i\left(\left\{\Gamma_{\alpha\beta}\right\}\right) = 0,\ \left\{\Gamma_{\alpha\beta}\right\} = \Gamma_{11}, \Gamma_{12}, \ldots \Gamma_{nn}, \quad i = 1,2, \ldots p \tag{18}$$

where $\Gamma_{\alpha\beta}^{*i}$ are the results of noisy measurements, and $\Gamma_{\alpha\beta}^{i}$ are the corrected values of the corresponding components. The idea of the approach is that the corrected values $\Gamma_{\alpha\beta}^{i}$ must satisfy the constraints (18), and at the same time, must be as close as possible (in terms of the least square) to the results of measurements $\Gamma_{\alpha\beta}^{*i}$. Introducing the Lagrange multipliers λ_i, one faces the following problem: find corrected values $\Gamma_{\alpha\beta}^{i}$ of the gravity gradient tensor in n^3 points based upon noisy measurements $\Gamma_{\alpha\beta}^{*i}$ by minimizing the following function

$$\tilde{M} = \frac{1}{2}\sum_{i-1}^{n^3}\sum_{sk}(\Gamma_{\alpha\beta}^{*i} - \Gamma_{\alpha\beta}^{i})^2 + \sum_{i=1}^{p}\lambda_i f_i\left(\left\{\Gamma_{\alpha\beta}\right\}\right) \tag{19}$$

It should be noticed that, strictly speaking, some of the constraints (18) are differential; however, since the values of $\Gamma_{\alpha\beta}^{i}$ are defined only in discrete n^3 points, it is reasonable to represent the derivatives $\partial\Gamma_{\alpha\beta}^{i}$ via their values at the corresponding discrete points using linear interpolation. After that, all the constraints become algebraic. The minimum of $\tilde{M}$ in equation (19) leads to a system of linear algebraic equations with respect to $\Gamma_{\alpha\beta}^{i}$ following from the conditions

$$\frac{\partial \tilde{M}}{\partial \Gamma_{\alpha\beta}^{i}} = 0 \tag{20}$$

4. ERROR MINIMIZATION BY REDUNDANT MEASUREMENT

In this section we will develop a methodology for reducing the total error of noisy measurements using redundant measurement following the ideas presented in the previous section. For the purpose of proof-of-concept, we will restrict ourselves by the case of three points on a plane x_1^0, x_2^0, x_3^0. Thus, since the gravimeter measures components of the tensor-gradient $\Gamma_{\alpha\beta}$, one can take advantage of the fact that these components are not independent because they must satisfy the integrability constraints (8), (11) and the zero sink-source constraint (13). Indeed, suppose that the points of measurements are located on (x, y)-plane, with the coordinate lines x_1 and x_2. We will consider only three points (i, j); $(i+1, j)$ and $(i, j+1)$ in order to illustrate the proposed approach. For simplicity, we will assume that the symmetry of the gravity gradient tensor has been already imposed, i.e., we will consider only six components of the tensor $\Gamma_{\alpha\beta}$. Hence, the problem will be stated as follows: Suppose 3x6 components $\Gamma_{\alpha\beta}^{*ij}, \Gamma_{\alpha\beta}^{*i+1,j}$ and $\Gamma_{\alpha\beta}^{*i,j+1}$ of the tensors $\Gamma_{\alpha\beta}^{*}$ at three points are given as a result of measurements. Assume that these measurements have errors, i.e.,

$$\Gamma_{\alpha\beta}^{*ij} = \Gamma_{\alpha\beta}^{oi,j} + \Delta\Gamma_{\alpha\beta}^{i,j}, \quad \text{etc} \tag{21}$$

Let us minimize these errors

$$\sum_{i,j\alpha,\beta} \left(\Delta\Gamma_{\alpha\beta}^{ij}\right)^2 \to \min \tag{22}$$

using the compatibility equations (8) and (11) as constraints imposed upon the exact values $\Gamma_{\alpha\beta}^{oij}$, $\Gamma_{\alpha\beta}^{oi+1j}$,... etc. First of all, one has to realize that out of 9 constraints of equation (11), only three are applicable, since we are dealing with the compatibility on the surface (x, y). Ignoring the curvature of this surface, one can use only the constraints which follow from the third row in the matrix (11), i.e.,

$$\partial_1\Gamma_{12} - \partial_2\Gamma_{11} = 0, \quad \partial_1\Gamma_{22} - \partial_2\Gamma_{12} = 0, \quad \partial_1\Gamma_{23} - \partial_2\Gamma_{13} = 0 \tag{23}$$

However, in order to apply these differential constraints to the measurement data, one should present the latter in an analytical form. Using a simple linear interpolation, one can write

$$\Gamma_{\alpha\beta} = \Gamma_{\alpha\beta}^{i,j} + \left(\Gamma_{\alpha\beta}^{i+1,j} - \Gamma_{\alpha\beta}^{i,j}\right)x_1 + \left(\Gamma_{\alpha\beta}^{i,j+1} - \Gamma_{\alpha\beta}^{ij}\right) x_2 \tag{24}$$

Then the constraints (23) take the form

$$\begin{aligned}
\Gamma_{12}^{i+1,j} - \Gamma_{12}^{i,j} - \Gamma_{11}^{i,j+1} + \Gamma_{11}^{i,j} &= 0\\
\Gamma_{22}^{i+1,j} - \Gamma_{22}^{i,j} - \Gamma_{12}^{i,j+1} + \Gamma_{12}^{i,j} &= 0\\
\Gamma_{23}^{i+1,j} - \Gamma_{23}^{i,j} - \Gamma_{13}^{i,j+1} + \Gamma_{13}^{i,j} &= 0
\end{aligned} \tag{25}$$

These equations should be supplemented by the algebraic constraints (13) written for each point:

$$\begin{aligned}
\Gamma_{11}^{i,j} + \Gamma_{22}^{i,j} + \Gamma_{33}^{i,j} &= 0\\
\Gamma_{11}^{i+1,j} + \Gamma_{22}^{i+1,j} + \Gamma_{33}^{i+1,j} &= 0\\
\Gamma_{11}^{i,j+1} + \Gamma_{22}^{i,j+1} + \Gamma_{33}^{i,j+1} &= 0
\end{aligned} \tag{26}$$

So, now we need to update the measurements $\Gamma_{\alpha\beta}^{*}$ so that after substituting them into the equations (25), (26), the mismatch will be minimal. In other words, we have to find such new updated values of measurements $\Gamma_{\alpha\beta}$ which minimize the sum

$$M = \frac{1}{2}\sum_{sk}\left[\left(\Gamma_{\alpha\beta}^{*i,j} - \Gamma_{\alpha\beta}^{i,j}\right)^2 + \left(\Gamma_{\alpha\beta}^{*i+1,j} - \Gamma_{\alpha\beta}^{i+1,j}\right)^2 + \left(\Gamma_{\alpha\beta}^{*i,j+1} - \Gamma_{\alpha\beta}^{i,j+1}\right)^2\right] \to \min \tag{27}$$

subject to the constraints (25) and (26). Introducing the Lagrange multipliers for these constraints as: $\lambda_1, \ldots, \lambda_6$ one can rewrite equation (27) in the form

$$
\begin{aligned}
M = \frac{1}{2}\sum_{sk}\Big[&\left(\Gamma^{*i,j}_{12}-\Gamma^{i,j}_{12}\right)^2+\left(\Gamma^{*i+1,j}_{12}-\Gamma^{i+1,j}_{12}\right)^2+\left(\Gamma^{*i,j+1}_{12}-\Gamma^{i,j+1}_{12}\right)^2\Big]+\\
&+\lambda_1\left(\Gamma^{i+1,j}_{12}-\Gamma^{i,j}_{12}-\Gamma^{i,j+1}_{11}+\Gamma^{i,j}_{11}\right)+\lambda_2\left(\Gamma^{i+1,j}_{22}-\Gamma^{i,j}_{22}-\Gamma^{i,j+1}_{12}-\Gamma^{i,j}_{12}\right)+\\
&+\lambda_3\left(\Gamma^{i+1,j}_{23}-\Gamma^{i,j}_{23}-\Gamma^{i,j+1}_{13}+\Gamma^{i,j}_{13}\right)+\lambda_4\left(\Gamma^{i,j}_{11}+\Gamma^{i,j}_{22}+\Gamma^{i,j}_{33}\right)+\\
&+\lambda_5\left(\Gamma^{i+1,j}_{11}+\Gamma^{i+1,j}_{22}+\Gamma^{i+1,j}_{33}\right)+\lambda_6\left(\Gamma^{i,j+1}_{11}+\Gamma^{i,j+1}_{22}+\Gamma^{i,j+1}_{33}\right)\to \min
\end{aligned}
\tag{28}
$$

The minimum of M requires that

$$
\frac{\partial M}{\partial \Gamma^{i,j}_{\alpha\beta}}=0,\ \frac{\partial M}{\partial \Gamma^{i+1,j}_{\alpha\beta}}=0,\ \frac{\partial M}{\partial \Gamma^{i,j+1}_{\alpha\beta}}=0 \tag{29}
$$

where $\alpha,\beta=1,2,3$. The values of updated measurements $\Gamma_{\alpha\beta}$ must satisfy the following system of linear equations

$$
\begin{array}{lll}
\Gamma^{i,j}_{11}+\lambda_1+\lambda_4=\Gamma^{*i,j}_{11}; & \Gamma^{i+1,j}_{11}+\lambda_5=\Gamma^{*i+1,j}_{11}; & \Gamma^{i,j+1}_{11}-\lambda_1+\lambda_6=\Gamma^{*i,j+1}_{11}\\
\Gamma^{i,j}_{12}-\lambda_1+\lambda_2=\Gamma^{*i,j}_{12}; & \Gamma^{i+1,j}_{12}+\lambda_1=\Gamma^{*i=1,j}_{12}; & \Gamma^{i,j+1}_{12}-\lambda_2=\Gamma^{*i,j+1}_{12}\\
\Gamma^{i,j}_{13}+\lambda_3=\Gamma^{*i,j}_{13}; & \Gamma^{i+1,j}_{13}=\Gamma^{*i+1,j}_{13}; & \Gamma^{i,j+1}_{13}-\lambda_3=\Gamma^{*i,j+1}_{13}\\
\Gamma^{i,j}_{22}-\lambda_2+\lambda_4=\Gamma^{*i,j}_{22}; & \Gamma^{i+1,j}_{22}+\lambda_2+\lambda_5=\Gamma^{*i+1,j}_{22}; & \Gamma^{i,j+1}_{22}+\lambda_6=\Gamma^{*i,j+1}_{22}\\
\Gamma^{i,j}_{23}-\lambda_3=\Gamma^{*i,j}_{23}; & \Gamma^{i+1,j}_{23}+\lambda_3=\Gamma^{*i+1,j}_{23}; & \Gamma^{i,j+1}_{23}=\Gamma^{*i,j+1}_{23}\\
\Gamma^{i,j}_{33}+\lambda_4=\Gamma^{*i,j}_{33}; & \Gamma^{i+1,j}_{33}+\lambda_5=\Gamma^{*i+1,j}_{33}; & \Gamma^{i,j+1}_{33}+\lambda_6=\Gamma^{*i,j+1}_{33}
\end{array}
\tag{30}
$$

As follows from equations (30), two components of the gravity gradient, namely, $\Gamma^{i+1,j}_{13}$ and $\Gamma^{i,j+1}_{23}$, are not updated: they are equal to the measured values $\Gamma^{*i+1,j}_{13}$ and $\Gamma^{*i,j+1}_{23}$, i.e. the 9-th and 17-th equations out of 18 equations (30), are already solved, and they can be excluded from the further consideration. Such an asymmetry with respect to different components of the gravity gradient is caused by the asymmetry of the problem itself. Firstly, there is no constraint in the direction of x_3 coordinate. Secondly, among three points of measurement: (i,j) ; $(i+1,j)$ and $(i,j+1)$, the first point plays the role of a center of linear interpolation.

Thus, now the rest 16 equations in (30) should be solved simultaneously with the compatibility equations (25) and (26). The system can be written in the following matrix form

$$D\Im = \Im^* \tag{31}$$

where D is (22x22) matrix and $\Im, \Im^*$ are 22 element vectors consisted of components of gravity gradient tensor. The solution to the problem is reduced to inversion of the matrix D

$$\Im = D^{-1}\Im^* \tag{32}$$

The proposed methodology can be generalized to an arbitrary number of points of measurements without any significant changes.

5. NUMERICAL EXPERIMENT

In order to illustrate the proposed methodology, we will perform some computer simulations.
First, let us create synthetic data of measurement for gravity gradient tensor. For this purpose, we take a point mass object, and construct gravity gradient data. We will call these data theoretical gravity gradient data. In terms of the equation (21) they will correspond to the values $\Gamma^{0^{i,j}}_{\alpha\beta}$, $\Gamma^{0^{i+1,j}}_{\alpha\beta}$ and $\Gamma^{0^{i,j+1}}_{\alpha\beta}$.
Second, we create errors for each component with $\Delta\Gamma^{i,j}_{\alpha\beta}$, $\Delta\Gamma^{i+1,j}_{\alpha\beta}$ and $\Delta\Gamma^{i,j+1}_{\alpha\beta}$ assuming that $\Delta\Gamma_{\alpha\beta} = \Delta\Gamma_{\alpha\beta}$, $\Delta\Gamma^{i+1,j}_{13} = 0$ and $\Delta\Gamma^{i+1,j}_{23} = 0$. The values of the error should be randomly distributed over the 16 components, while their absolute values should be of the same order as the expected errors of the gradiometer.
Adding the errors to the theoretical values, we create synthetical measurement data

$$\begin{aligned}
\Gamma^{*^{i,j}}_{\alpha\beta} &= \Gamma^{0^{i,j}}_{\alpha\beta} + \Delta\Gamma^{i,j}_{\alpha\beta} \\
\Gamma^{*^{i+1,j}}_{\alpha\beta} &= \Gamma^{0^{i+1,j}}_{\alpha\beta} + \Delta\Gamma^{i+1,j}_{\alpha\beta} \\
\Gamma^{*^{i,j+1}}_{\alpha\beta} &= \Gamma^{0^{i,j+1}}_{\alpha\beta} + \Delta\Gamma^{i,j+1}_{\alpha\beta}
\end{aligned} \tag{33}$$

or, utilizing the compressed notation

$$\Gamma_i^* = \Gamma_i^0 + \Delta\Gamma_i \tag{34}$$

where index i indicates the number of the corresponding components of the gravity gradient tensor (For example, in case of three nearest points on (x, y)-plane we have i=1,...,16). The accuracy of the measurements (34) can be characterized by the variance with respect to Γ_i^0

$$S_1^2 = (1/n)\sum_{i=1}^{n}\left(\Gamma_i^* - \Gamma_i^0\right)^2 \tag{35}$$

where $n = 16$ is the number of the considered components of the gravity gradient tensor. The next step of the proposed methodology is to construct vectors from the components of gravity gradient tensor for the nearest points and to update the values of the measurements Γ_i using the equation

$$\Im = D^{-1}\Im^* \tag{36}$$

As a result we obtain a new error

$$\Delta\Gamma_i = \Gamma_i - \Gamma_i^0 \tag{37}$$

and the variance

$$S_2^2 = (1/n)\sum_{i=1}^{n}\left(\Gamma_i - \Gamma_i^0\right)^2 \tag{38}$$

where $i = 1,..., n$, and $n = 16$.

The idea of the proposed application is to show that the total error of the updated measurements Γ_i is smaller than those of the original measurements Γ_i^*, i.e. that $S_2 < S_1$.

Computer simulation.

We start with the modeling of gravity gradient tensor $\Gamma_{\alpha\beta}$ (equation (3)) for point-like object with the mass $M = G^{-1}$, where G is gravitational constant at the distance 200 km from the source. The components $\Gamma_{\alpha\beta}$ are constructed on 500x500x50 grid with grid spacing 5000 and 3000 meters (Fig. 2). For computer analysis of the model we select 20x20x50 grid at the center of the larger one (Fig. 1) where the variation of gravity gradient components is maximal. We vary the components of gravity gradient tensor with random function f_{rand}. The amplitude of random function is taken as 3% of the maximal value of the gravity gradient tensor at each point of the small grid $\Delta\Gamma_{\alpha\beta} = f_{rand} \cdot \Gamma_{\alpha\beta}$. We assume here that synthetically modeled data are experimental ones and write the result as $\Gamma_{\alpha\beta}{}^{*} = \Gamma_{\alpha\beta} + \Delta\Gamma_{\alpha\beta}$. As the next step, we construct 16-component vector $\Im^{*}$ with Γ_i^{*} ($i = 1,...,16$) in the order, which is given by

$$\Im^{*} = \left[\Gamma_1^{*} \ldots \Gamma_{16}^{*} \; 0 \; 0 \; 0 \; 0 \; 0 \; 0\right] \tag{39}$$

and 22-component vector $\Im$ constructed with unknown elements Γ_i ($i = 1,...,16$) and λ_j ($j = 1,...,6$):

$$\Im = \left[\Gamma_1 \ldots \Gamma_{16} \; \lambda_1 \; \lambda_2 \; \lambda_3 \; \lambda_4 \; \lambda_5 \; \lambda_6\right] \tag{40}$$

Then (16x16)-component inversion matrix D^{-1} is found with constraints. After the inversion we obtain the matrix D^{-1}. For the analysis, we applied D^{-1} to the vector $\Im^{*}$ (39) in the small (20x20) region and found the components of vector $\Im$ (equation 40) for each point of this region as $\Im = D^{-1}\Im^{*}$.

For estimation of the modeled data we use the equations (35) and (38). The result is given on Fig. 3 for the grid spacing 3000 meters and Fig. 5 for the grid spacing 5000 meters. Standard deviations S_1 (marked with the boxes) and S_2 for each 400 points describe experimental and theoretical results for the computer simulations. As follows from Fig. 3 and Fig. 5, the error S_2 is always smaller then S_1. For the quantitative estimation of the improvement we use the ratio $(S_1 - S_2)/S_2$ for each point on small grid. These results are given on Fig. 4 for the grid spacing 3000 meters, and on Fig. 6 for the grid spacing 5000 meters.

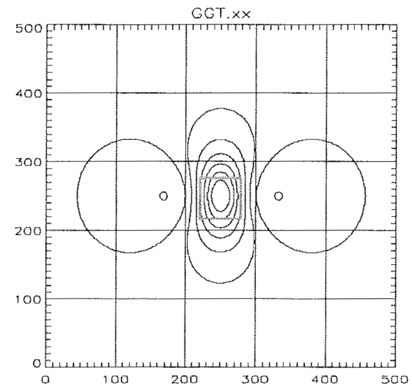


Fig. 1 Profiles of Γ_{xx} component of gravity gradient tensor. A small grid of the size 20x20x50 is selected in the area of the maximal variations of gravity gradient.

6. CONCLUSION

Thus, it has been proposed a new methodology for gravity gradient data improvement based upon combining measurement data and the integrability constraints. Two alternative strategies were proposed: reconstruction of the gravitational field based upon the minimal number of measurements, and reconstruction of the total error of noisy measurements exploiting the redundancy of the measurements of the gravity gradient tensor. A general formalism illustrated by a computer experiment has been developed. Computer analysis of output data showed that the developed formalism allows us to analytically improve the experimental gravity gradient data by up to 25%.

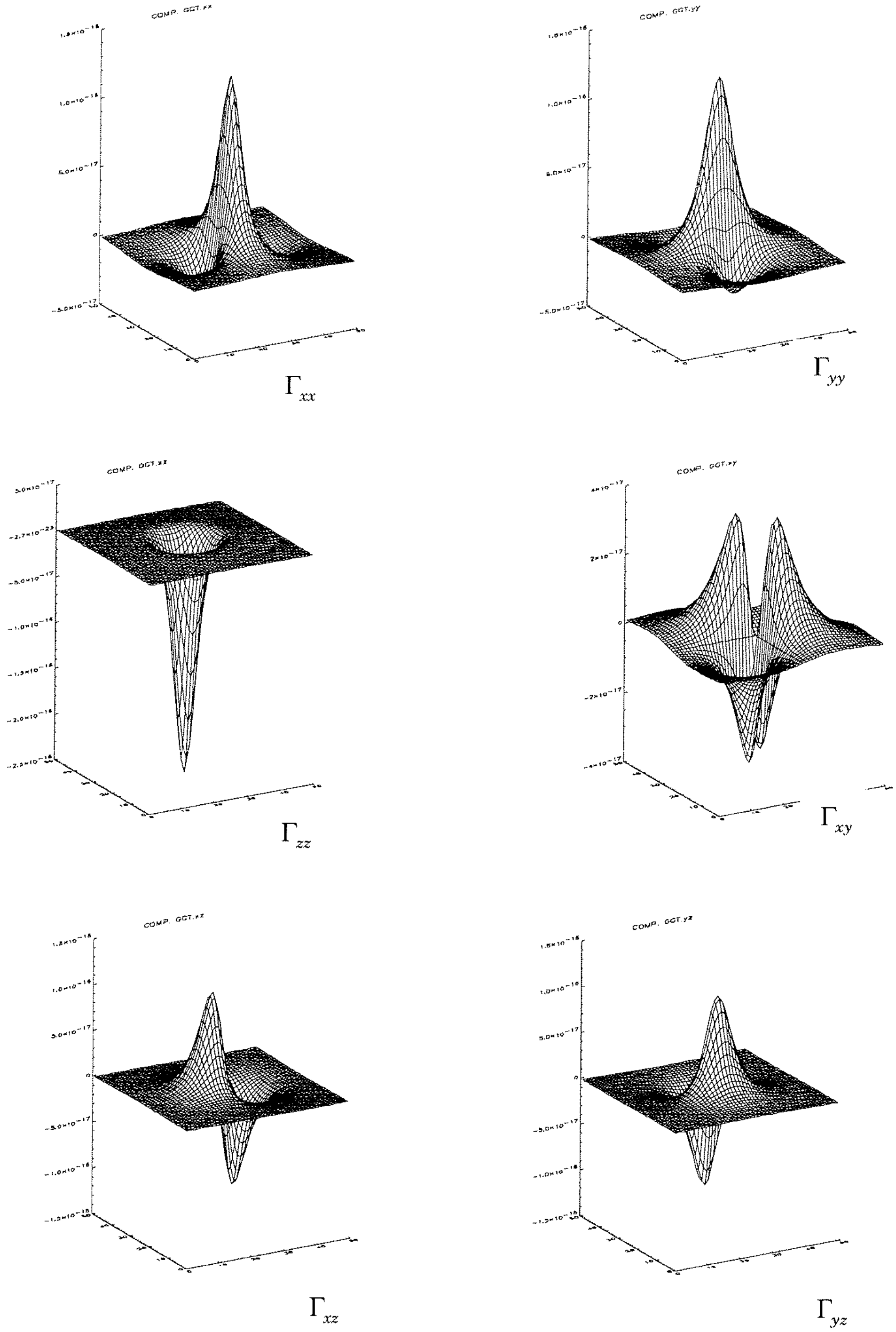
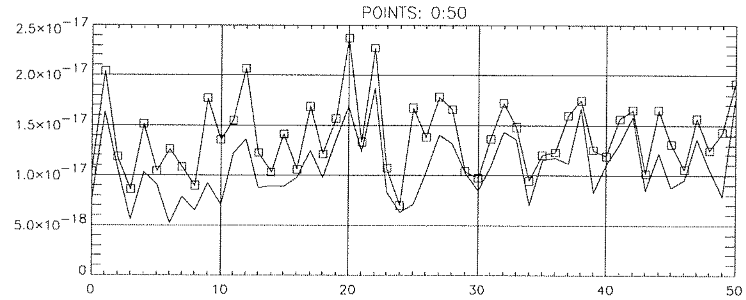


Fig. 2 *Surfaces for the components of gravity gradient tensor with 5000 meter grid spacing.*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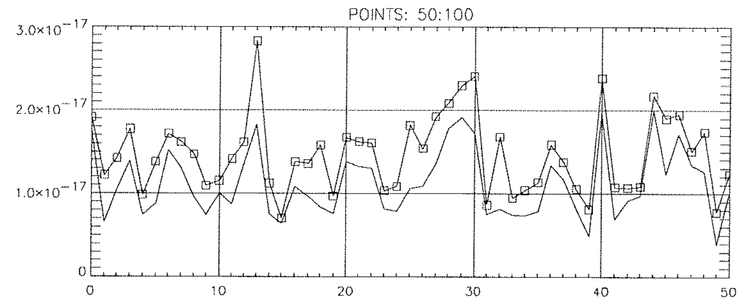
2.0×10^{-17}
1.5×10^{-17}
1.0×10^{-17}
5.0×10^{-18}
0
0 10 20 30 40 50


POINTS: 50:100
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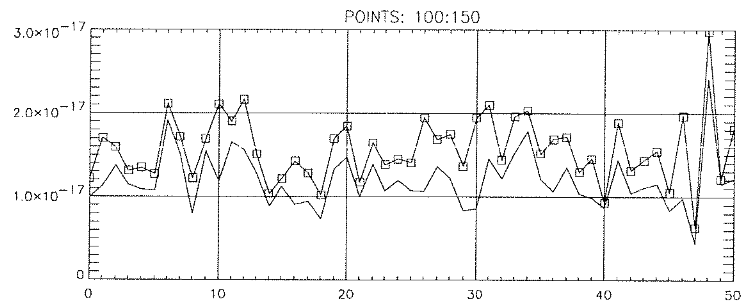
0
0 10 20 30 40 50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1.0×10^{-17}
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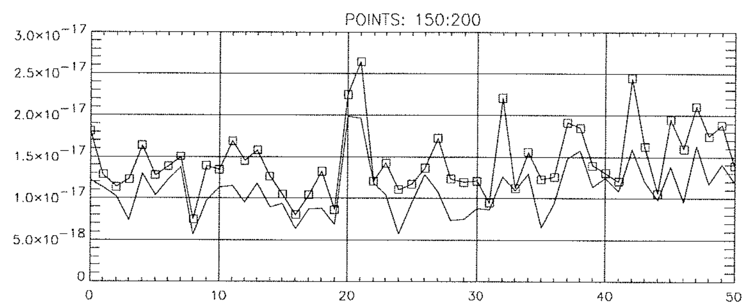
POINTS: 150:200
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1.0×10^{-17}
5.0×10^{-18}
0
0 10 20 30 40 50

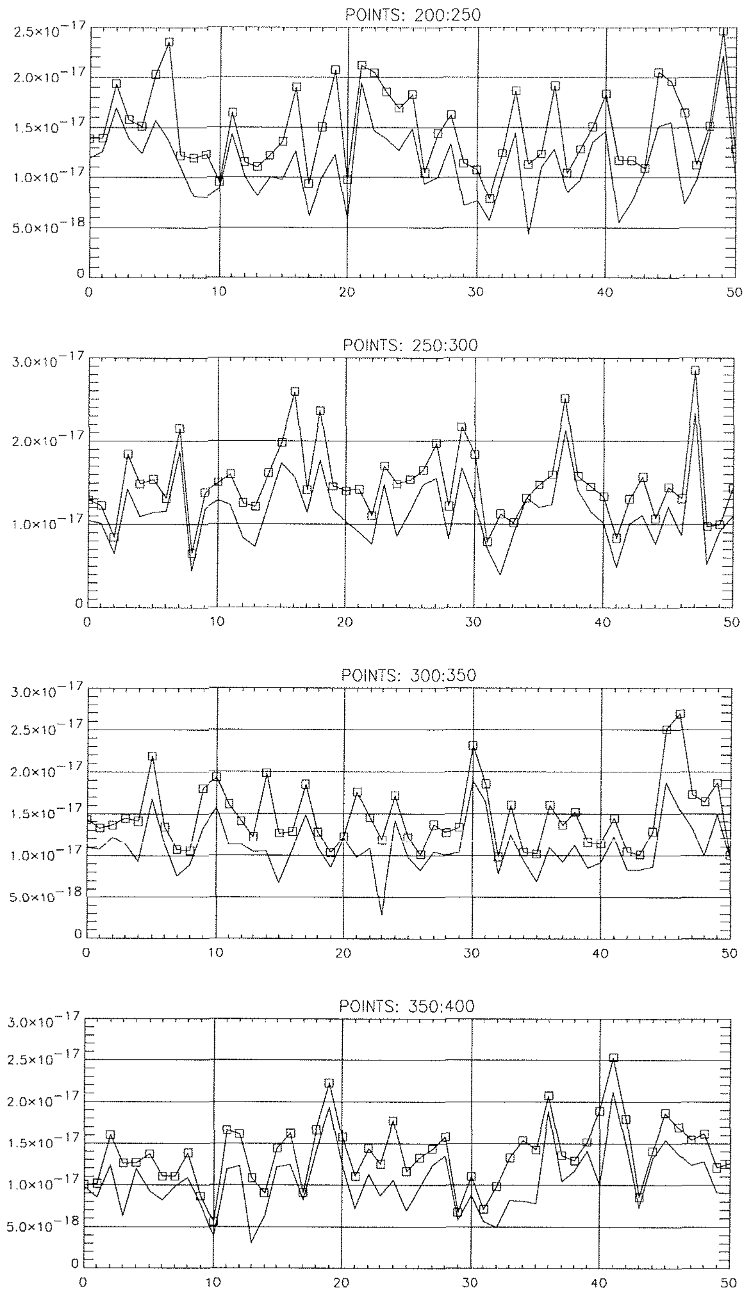
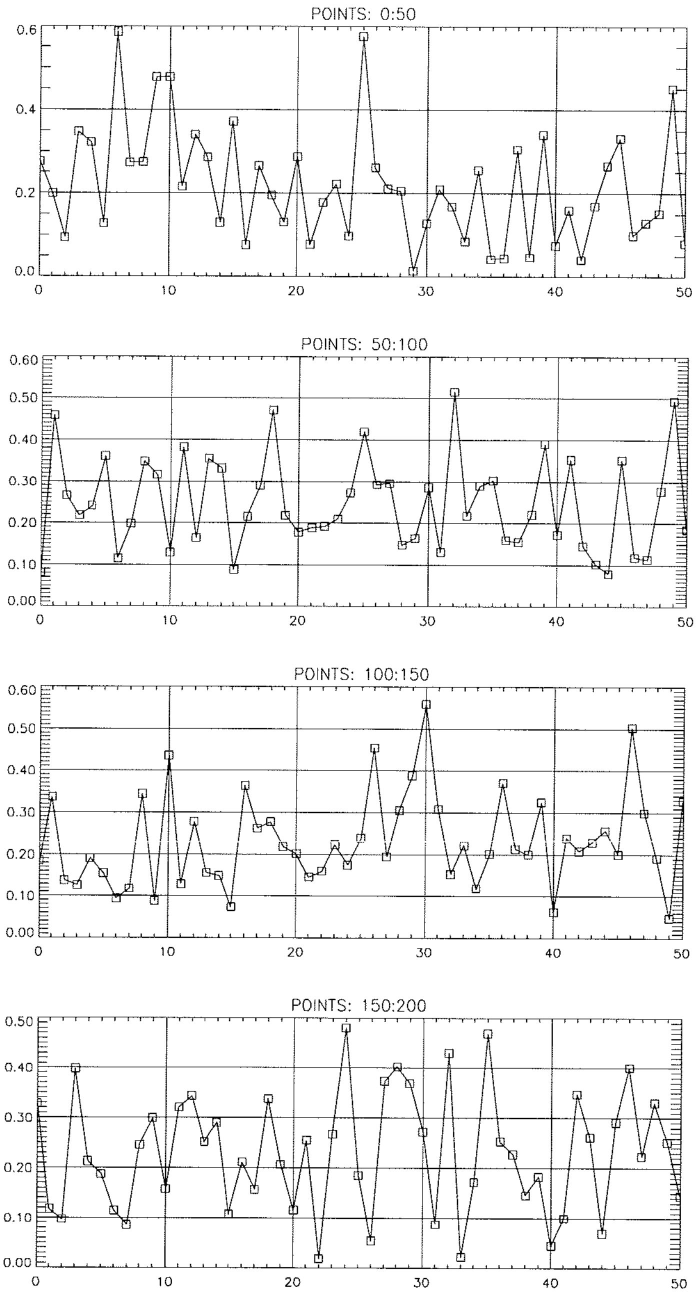


Fig. 3 *Experimental S_1 (boxed) and theoretical S_2 curves for standard deviations show the improvement of measured gravity gradient data for 3000 meter grid spacing*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POINTS: 50:100
POINTS: 100:150
POINTS: 150:200

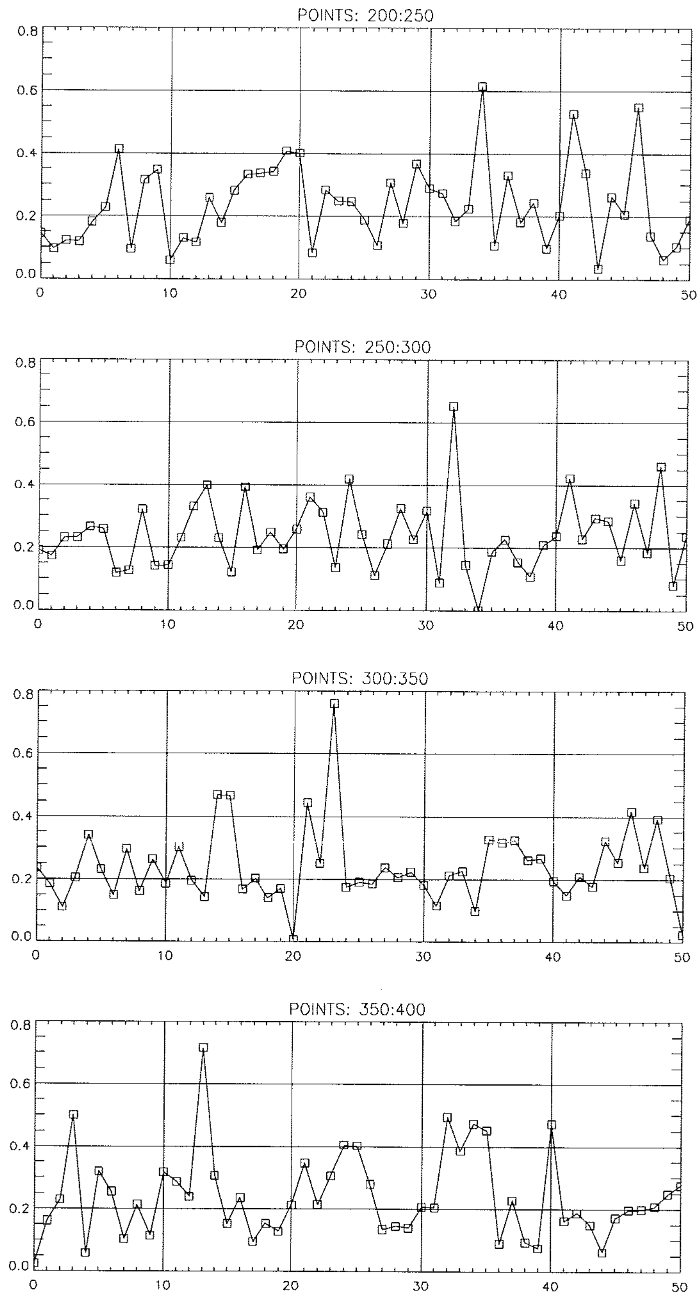
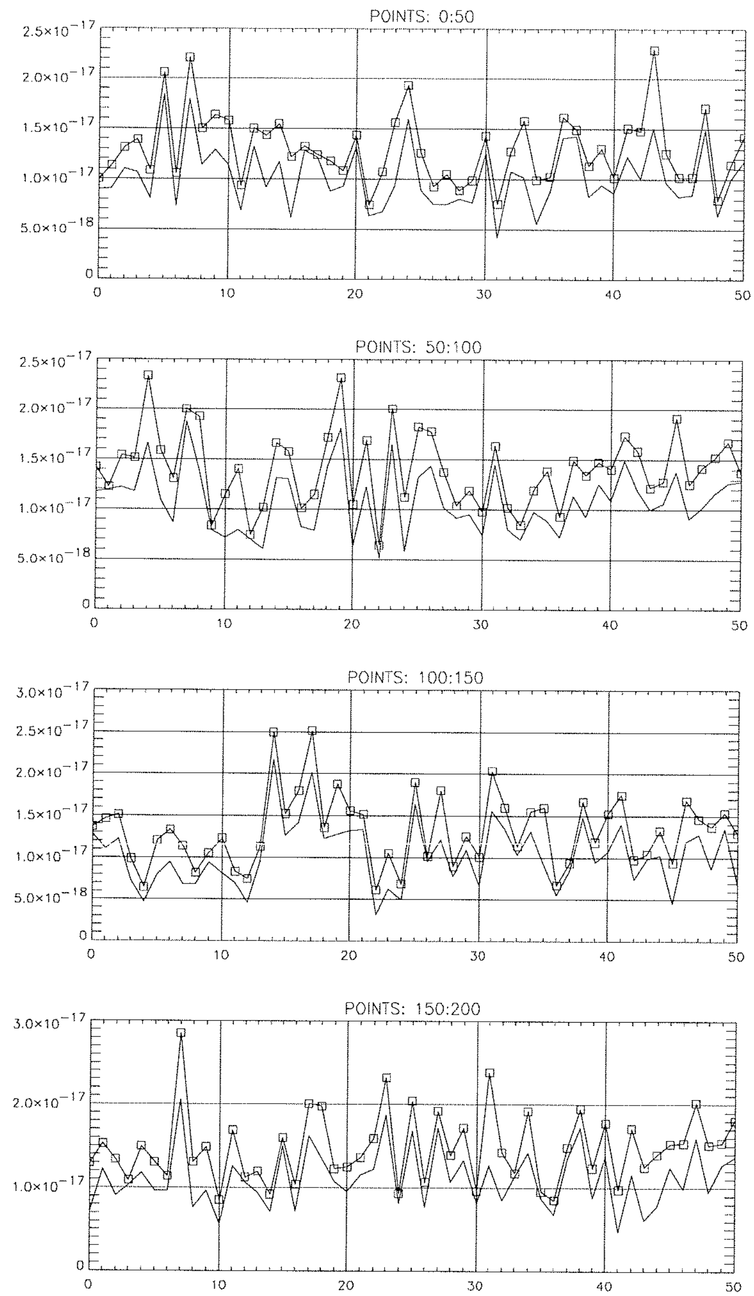


Fig.4 *The improvement* $[(S_1\text{-}S_2)/S_2] \times 100\%$ *of the measurement of gravity gradient tensor for 5000 meter grid spacing*


POINTS: 0:50
POINTS: 50:100
POINTS: 100:150
POINTS: 150:200
2.5×10^-17
2.0×10^-17
1.5×10^-17
1.0×10^-17
5.0×10^-18
3.0×10^-17
0
10
20
30
40
50

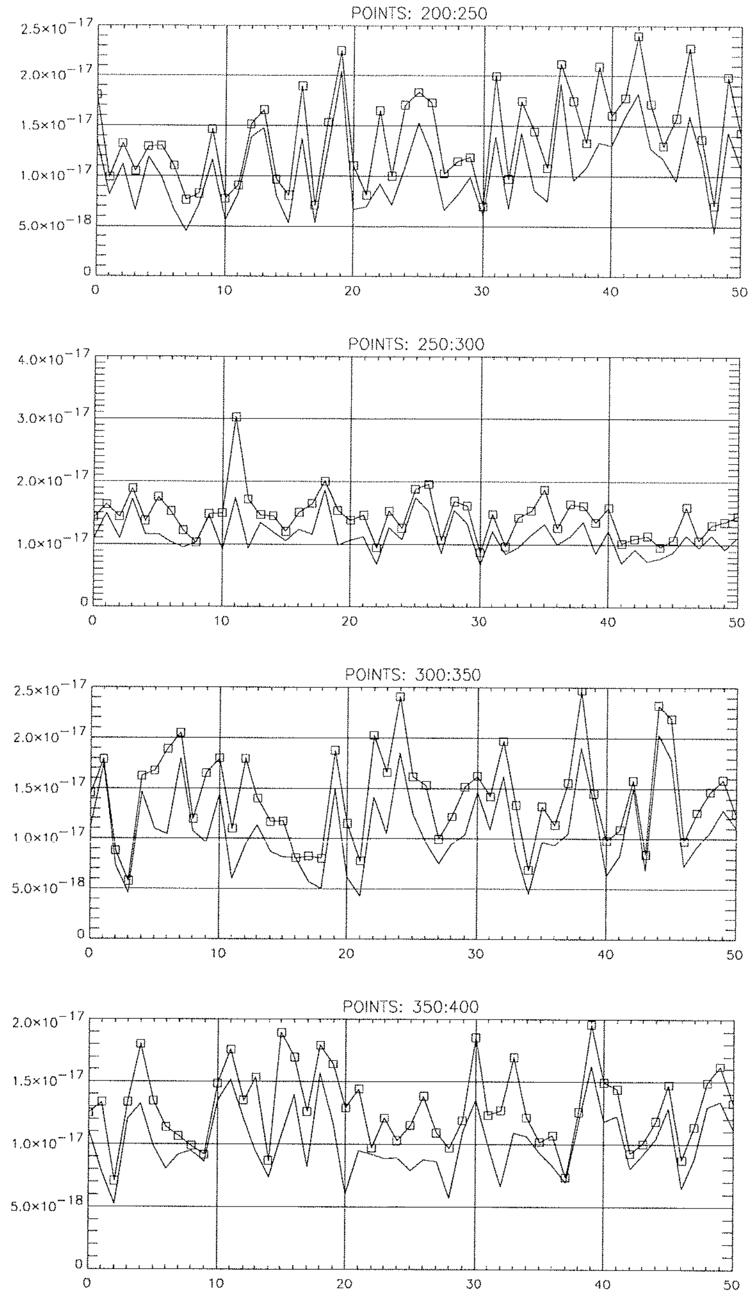
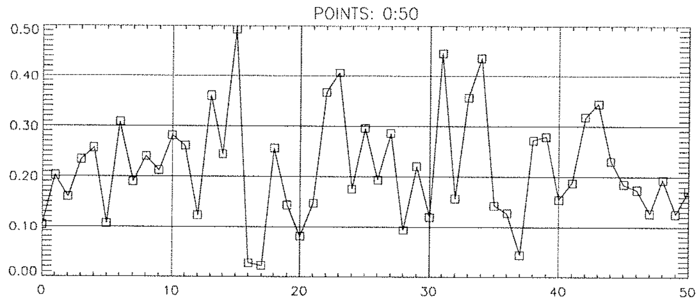


Fig. 5 *Experimental S_1 (boxed) and theoretical S_2 curves for standard deviations show the improvement of measured gravity gradient data for 5000 meter grid spacing*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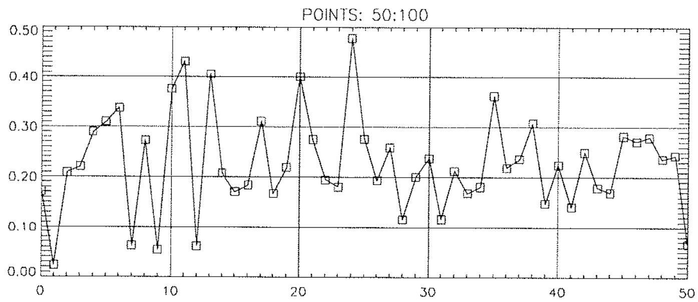
0.30
0.20
0.10
0.00
0
10
20
30
40
50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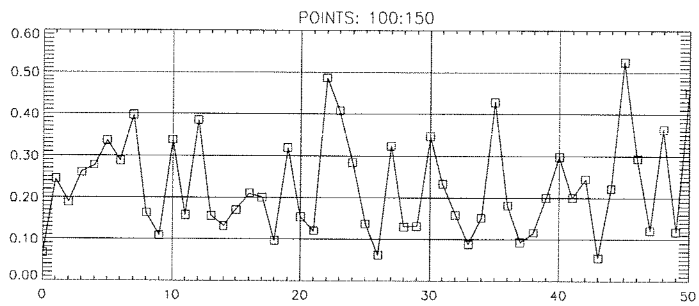
0.20
0.10
0.00
0
10
20
30
40
50


POINTS: 100:150
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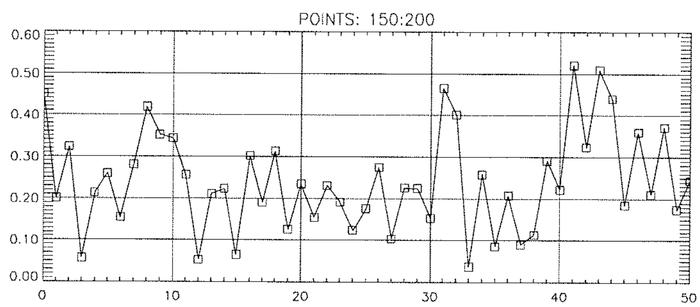
0.10
0.00
0
10
20
30
40
50


POINTS: 150:200
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0.30
0.20
0.10
0.00
0
10
20
30
40
50

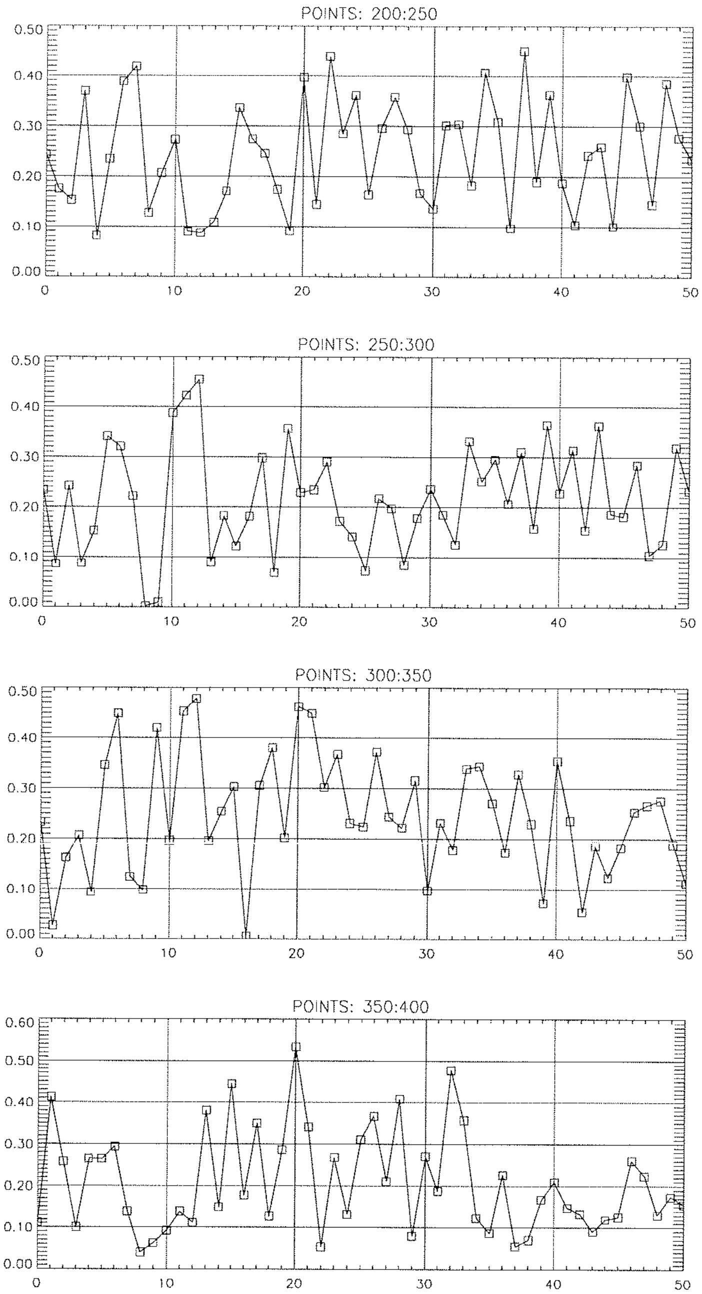

Fig. 6 *The improvement $[(S_1\text{-}S_2)/S_2]$ x 100% of the measurement of gravity gradient tensor for 5000 meter grid spacing*

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